Error Bounds in Taylor Approximations

Friday, April 21, 2023 9:35 AM

thm: (error bound in taylor approximation) $f: \mathbb{R} \to \mathbb{R}$ differentiate @ $x = a$

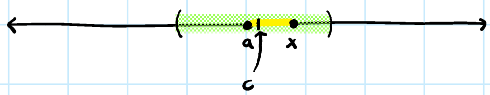
then,

$$f(x) = \overbrace{f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}^{\text{polynomial of degree } n} + R_{n+1}$$

error → part missing

where error function is $\underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{next term}}$ with c between a & x

bounded by



$\frac{f^{(n+1)}}{(n+1)!}$ evaluated somewhere in —

* x closer & closer to a → value of exact # missing from approximation bounded by c *

ex 1) give $\cos(0.1)$ with error $\leq 10^{-3}$

$$\cos(0.1) = \underbrace{1 - \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24} - \frac{(0.1)^6}{720}} + \frac{(0.1)^8}{8!} - \frac{(0.1)^{10}}{10!} + \dots$$

taylor $\cos(x)$ @ $x=0$ & $R = \infty$

this truncation has error bounded by ...

$c \in (0, 0.1)$

max value $\cos(c)$ can go to

$$\left| \frac{\cos^{(7)}(c)}{720 \cdot 7} \right| \leq \frac{1}{720 \cdot 7} \approx 1.98 \times 10^{-4} \leq 10^{-3} \checkmark$$

next term after truncation

$$\lim_{n \to \infty} \left| \frac{\frac{x^{2(n+1)}}{(2(n+1))!}}{\frac{x^{2n}}{2n!}} \right| = \frac{x}{(n+1)} = 0 \quad R = \infty$$